Asynchronous Control of Nonlinear Markov Jump Systems With Uncertainties Using Interval Type-2 Polynomial Fuzzy Approach

Zhaowen Xu, Zheng-Guang Wu, Haoyi Que, *Member, IEEE*, and Peng Jiang

***Abstract*—The focus of this article is to address the asynchronous control issue for a class of nonlinear Markov jump systems with parameter uncertainties. They are represented by the interval type-2 polynomial fuzzy model. A hidden Markov model is utilized to describe the asynchronous behavior between the system mode and the controller mode in a quantitative manner. This is achieved by using a joint random process, which is presented in a concise format and encompasses both spontaneous and simultaneous jumps. Further to facilitate the design flexibility and low implementation burden, the structure of the asynchronous polynomial fuzzy controller is formed based on the imperfect premise matching scheme. Facing both mismatched modes and mismatched premise variables, a joint Markov chain-based membership-function-dependent stability criteria is established by utilizing the Lyapunov–Krasovskii theory and the $\mathcal{H}_\infty$ performance analysis is then conducted using a sum-of-square approach. The polynomial feedback gain parameters can be readily solved by a sum of squares optimization toolbox SOSTOOLS. Finally, a numerical example is used to validate the effectiveness of our obtained results and the application potential is verified by a single-link robot arm model.**

***Index Terms*—Asynchronous control, interval type-2 (IT2) polynomial fuzzy model, nonlinear Markov jump system (MJS).**

I. Introduction

The Markov jump system (MJS) has gained its reputation for well modeling system with abrupt changes in system structure, sensor failures, and stochastic perturbations and has been successfully applied in describing the traits of communication channel, circuit system failures, and so on [1], [2]. By far much progress has been made in its stability analysis and control synthesis. For example, $\mathcal{H}_\infty$ control [3], event-triggered control [4], sliding-mode control [5], observer-based control [6], etc., have been solved with or without time delays or other disturbances.

As one of the frequently encountered issues in the control theory of MJSs, asynchronous control has been extensively studied in recent years. The network-induced communication constraints inevitably interfere with accurately identifying the mode information of the considered plant, degrading the control performance. The research work in [7] has adopted a nonhomogeneous Markov chain to describe this scenario and presented an linear matrix inequality (LMI)-based filter design scheme. In [8], [9], [10], and [11], a hidden Markov model has been introduced to establish a relationship between the mode information of a system and that of a designed controller. This approach has proven to be effective and has gained popularity due to its simple structure. For the nonlinear case, by contrast, the results obtained are not as abundant as in linear MJSs and usually not in closed forms, which motivates this work.

Thanks to Takagi–Sugeno (T–S) fuzzy theory, which offers a general framework for system analysis and control synthesis, the nonlinear part can be linearized by grouping a set of weighted linear descriptions [12], [13], [14], [15]. Most of the issues on the nonlinear Makov jump system are then discussed by utilizing fuzzy system theory [16], [17], [18], [19], [20]. The results are presented in terms of LMIs and can be solved numerically by using convex programming techniques. For example, the research in [21] has addressed the output tracking issue in MJSs with packet dropouts with the aid of a T–S fuzzy approach. The quantization control issue has been concerned in [22] and controller gains are designed on the fuzzy-basis-dependent Lyapunov function. However, it can be seen that in this routine, the conditions derived are independent of the membership functions (MFs), which means that the obtained results are applicable for any fuzzy model-based systems as long as they contain the same subsystems regardless of MFs. Since the MFs preserve the intrinsic nonlinear nature of the original system, the conservatism is apparent if they are dropped out. Therefore, it is of great significance for them to be integrated into the analysis and synthesis.

In addition, among the abovementioned works on fuzzy MJSs, the MFs are mainly based on the type-1 fuzzy set. For systems with parameter uncertainties, type-1 fuzzy set is not applicable

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since it contains only certain information. Then, by introducing the concept of footprint of uncertainty (FOU), type-1 fuzzy set is rendered into type-2 fuzzy set and can be called fuzzy–fuzzy set [23], which incorporates parameter uncertainties into the MFs and, thus, is capable of handling system uncertainties. Since the general type-2 MFs bring huge complexity embedded in the FOU, the widely used in applications is the interval type-2 fuzzy system model, where the upper and lower bounds of the MFs are adopted to reproduce them. Relavent results have been frequently reported recently (see [17], [24], [25], [26], [27] and the reference therein). To further the research in the interval type-2 fuzzy system, in [28], a membership-function-dependent (MFD) approach has been proposed to conduct the stability analysis and controller design to reduce the conservatism in conventional method. It is followed up by many works where the topics cover sampled-data control [24], $l_2 - l_\infty$ control with delays [29], and so on. Moreover, in [30], a polynomial-based modeling and analysis framework has been proposed by utilizing a sum-of-squares (SOS) approach. This can be efficiently solved by a third party MATLAB toolbox SOSTOOLS [31]. As the order of the polynomial increases, the global approximation ability goes strong, which means that the number of fuzzy rules can be greatly reduced.

Therefore, from the perspective of modeling accuracy, computation complexity, and result conservatism, when we are dealing with the control problem on the nonlinear system with Markov jump parameters and uncertainties, the idea with priority is to transform it into an IT2 polynomial fuzzy MJS. However, the methodologies for this case have been rarely discussed, which needs to be further addressed.

In this work, the asynchronous control problem is investigated for nonlinear MJSs with parameter uncertainties. Based on the IT2 polynomial fuzzy model-based approach, the system is approximated through polynomial fuzzy equations. The IT2 MFs are utilized to characterize system uncertainties. The controller is constructed such that it does not share the system mode information and the MFs for implementation purposes. Then, under the framework of Lyapunov–krasoviskii theory, stability conditions with guaranteed $\mathcal{H}_\infty$ performance are established in terms of SOS by matrix manipulation and piecewise linear function approximation method. The fuzzy asynchronous controller gain matrix then can be directly obtained by SOSTOOLS. Our main contributions are as the following three aspects.

1) A systematic way of discussing the asynchronous control problem for nonlinear MJSs with parameter uncertainties has been presented and discussed based on polynomial fuzz model approach.
2) Compared with existing results on fuzzy MJSs, here in this article the polynomial fuzzy model are adopted to give a better approximation to the original nonlinear MJSs with less fuzzy rules. Then, the transitions of the hidden Markov model are modified and presented in a way such that it can be utilized with the MFD method to give elegant results with less conservatism.
3) The final conditions are presented in the form of SOS, which, if feasible, can be readily checked by SOSTOOLS.

II. Preliminaries and Problem Formulation

We raise in this article the asynchronous control problem for nonlinear MJSs. To this end, we first introduce a hidden Markov model $\mathbf{H}(t) = (\mathbf{r}(t), \hat{\mathbf{r}}(t))$ where $\mathbf{r}(t)$ stands for the hidden state of the system mode taking values in $\mathcal{N} := \{1, 2, \ldots, N\}$ and the observation state $\hat{\mathbf{r}}(t)$ refers to the controller mode taking values in $\mathcal{S} := \{1, 2, \ldots, S\} \subseteq \mathcal{N}$. Suppose the joint process $\mathbf{H}(t)$ is homogeneous. It is defined in the set $\mathcal{V} = \mathcal{N} \times \mathcal{S}$. Let $\Pi = [\lambda_{pq}]$ be the transition matrix of $\mathbf{r}(t)$. Following the same line in [9], let $\alpha^\iota_{q\kappa}$ stand for the simultaneous jumps of $\hat{\mathbf{r}}(t)$ and $\mathbf{r}(t)$ and $\beta^p_{\iota\kappa}$ denotes the spontaneous jumps of $\hat{\mathbf{r}}(t)$. We can define the joint transition rate $\vartheta_{(p,\iota)(q,\kappa)}$ as

$$\vartheta_{(p,\iota)(q,\kappa)} = \begin{cases} \alpha^\iota_{q\kappa}\lambda_{pq}, & p \neq q, \kappa \in \mathcal{S} \\ \beta^p_{\iota\kappa}, & p = q, \iota \neq \kappa \\ \lambda_{pp} + \beta^p_{\iota\iota}, & p = q, \iota = \kappa \\ 0, & \text{otherwise} \end{cases} \tag{1}$$

where $\lambda_{pq} \geq 0 \,\forall i \neq j$, $\lambda_{pp} = -\sum_{p\neq q}\lambda_{pq}$, $\sum_{\kappa\in\mathcal{S}}\alpha^\iota_{q\kappa} = 1$, $\beta^p_{\iota\kappa} \geq 0 \,\forall \kappa \neq \iota$, and $\beta^p_{\iota\iota} = -\sum_{\kappa\neq\iota}\beta^p_{\iota\kappa}$. Thus, it can be checked that $-\vartheta_{(p,\iota)(p,\iota)} = -\lambda_{pp} - \beta^p_{\iota\iota} = \sum_{p\neq q}\lambda_{pq}\sum_{\kappa\in\mathcal{S}}\alpha^\iota_{q\kappa} + \sum_{\kappa\neq\iota}\beta^p_{\iota\kappa} = \sum_{p\neq q}\sum_{\kappa\in\mathcal{S}}\vartheta_{(p,\iota)(q,\kappa)} + \sum_{\kappa\neq\iota}\vartheta_{(p,\iota)(p,\kappa)} = \sum_{(p,\iota)\neq(q,\kappa)}\vartheta_{(p,\iota)(q,\kappa)}$. Furthermore, we have

$$\begin{aligned}&\mathbf{Pr}\{\mathbf{H}(t+\Delta)\} = (q,\kappa)|\mathbf{H}(t) = (p,\iota)\} \\ &\qquad = \begin{cases}\vartheta_{(p,\iota)(q,\kappa)}\Delta + o(\Delta) & (p,\iota) \neq (q,\kappa) \\ 1 + \vartheta_{(p,\iota)(q,\kappa)}\Delta + o(\Delta) & (p,\iota) = (q,\kappa).\end{cases}\end{aligned} \tag{2}$$

Remark 1: Some special cases of asynchronous phenomenon can be retrieved from the literature [32]. For example, when $\mathcal{S} = \{1\}$, $\beta^p_{\iota\kappa} = 0$, and $\alpha^1_{q1} = 1$, the controller is deterministic and not affected by the system mode, which is the mode-independent case. When $\mathcal{S} = \mathcal{N}$, $\beta^p_{\iota\kappa} = 0$, $\alpha^\iota_{qq} = 1$, and $\alpha^\iota_{q\kappa} = 0$ for $q \neq \kappa$ with invariant set $\mathcal{V} = \{(p,p) \in \mathcal{N} \times \mathcal{N}\}$, there are only simultaneous jumps to the same mode with equal transitions between $\mathbf{r}(t)$ and $\hat{\mathbf{r}}(t)$, which covers the mode-dependent case. It can be inferred that a larger $\beta^p_{\iota\kappa}$ means more frequent spontaneous jumps and a larger $\alpha^\iota_{q\kappa}$ for $q \neq \kappa$ means that system mode and controller mode are more likely to simultaneously jump but to different mode. For the cluster case where $S < N$, define a function $\mathscr{F} : \mathcal{N} \to \mathcal{S}$ where $\mathscr{F}(p) = \iota$. Then, let $\beta^p_{\iota\kappa} = 0$, $\alpha^\iota_{q\mathscr{F}(q)} = 1$, and we have that whenever the system mode transitions to q, the controller would have mode $\mathscr{F}(q)$. Therefore, by defining the values of $\beta^p_{\iota\kappa}$ and $\alpha^\iota_{q\kappa}$, the nonsynchronous degree can be quantitatively described.

Now a nonlinear MJS with parameter uncertainties is presented, which can be described by the following IT2 polynomial fuzzy model.

Rule i: IF $f_1(\mathbf{x}(t))$ is M^i_1 AND ... AND $f_\Phi(\mathbf{x}(t))$ is M^i_Φ, THEN

$$\dot{\mathbf{x}}(t) = \mathbf{A}^p_i(\mathbf{x}(t))\hat{\mathbf{x}}(\mathbf{x}(t)) + \mathbf{B}^p_i(\mathbf{x}(t))\mathbf{u}(t) + \mathbf{D}^p_i(\mathbf{x}(t))\mathbf{w}(t) \tag{3}$$

where $M^i_1, \ldots, M^i_\Phi$ are the fuzzy sets of rule i corresponding to the known function $f_\sigma(\mathbf{x}(t)), \sigma = 1, 2, \ldots, \Phi$ with $\Phi > 0$ being an integer. $\hat{\mathbf{x}}(\mathbf{x}(t)) \in \mathbb{R}^{\hat{n}}$ is a vector of monomials in $\mathbf{x}(t)$ where

$\mathbf{x}(t) \in \mathbb{R}^n$ denotes the system state vector. And $\mathbf{u}(t) \in \mathbb{R}^{n_u}$ is the input vector. $\mathbf{w}(t)$ infers the external disturbance belonging to $\mathcal{L}_2[0,+\infty)$. $\mathbf{A}_i^p(\mathbf{x}(t)), \mathbf{B}_i^p(\mathbf{x}(t)), \mathbf{D}_i^p(\mathbf{x}(t))$ are system matrices of monomials in $\mathbf{x}(t)$ and have appropriate dimensions for rule i when the system stays in mode p. The firing strength for the ith rule is represented by interval sets, i.e.,

$$\omega_i(\mathbf{x}(t)) = [\underline{\omega}_i(\mathbf{x}(t)), \overline{\omega}_i(\mathbf{x}(t))], i = 1, 2, \ldots, \varrho_1 \tag{4}$$

with

$$\underline{\omega}_i(\mathbf{x}(t)) = \prod_{\sigma=1}^{\Phi} \underline{\mu}_{M_\sigma^i}(f_\sigma(\mathbf{x}(t)))$$

$$\overline{\omega}_i(\mathbf{x}(t)) = \prod_{\sigma=1}^{\Phi} \overline{\mu}_{M_\sigma^i}(f_\sigma(\mathbf{x}(t)))$$

where $0 \le \underline{\mu}_{M_l^i}(f_l(\mathbf{x}(t))) \le \overline{\mu}_{M_l^i}(f_l(\mathbf{x}(t))) \le 1$, further resulting in $0 \le \underline{\omega}_i(\mathbf{x}(t)) \le \overline{\omega}_i(\mathbf{x}(t)) \le 1$. Then, the defuzzified form of the system dynamic can be written in a compact form

$$\mathcal{G}: \begin{cases} \dot{\mathbf{x}}(t) = \sum_{i=1}^{\varrho_1} \omega_i(\mathbf{x}(t))(\mathbf{A}_i^p(\mathbf{x}(t))\hat{\mathbf{x}}(\mathbf{x}(t)) \\ \qquad\qquad + \mathbf{B}_i^p(\mathbf{x}(t))\mathbf{u}(t) + \mathbf{D}_i^p(\mathbf{x}(t))\mathbf{w}(t)) \\ \mathbf{z}(t) = \mathbf{C}_i^p\hat{\mathbf{x}}(\mathbf{x}(t)). \end{cases} \tag{5}$$

For the control of MJSs, it is routine to assume that the mode information of the system is completely known, which is not practical and often limits its application. Hence, in our work, we construct an asynchronous nonlinear controller in the form of polynomials, which does not share the same mode with the system. Suppose it has ϱ_2 rules and based on the imperfect premise matching scheme it can be described by the following.

Rule j: IF $g_1(\mathbf{x}(t))$ is $\tilde{M}_1^j$ AND ... AND $g_{\tilde{\Phi}}(\mathbf{x}(t))$ is $\tilde{M}_{\tilde{\Phi}}^j$, THEN

$$\mathbf{u}(t) = K_j^\iota(\mathbf{x}(t))\hat{\mathbf{x}}(\mathbf{x}(t)) \tag{6}$$

in which $K_j^\iota(\mathbf{x}(t)) \in \mathbb{R}^{n_u \times \tilde{n}}$ denotes the asynchronous polynomial feedback gains. $g_{\tilde{\sigma}}(\mathbf{x}(t)), \tilde{\sigma} = 1, \ldots, \tilde{\Phi}$ and $\tilde{M}_1^j$, $j = 1, \ldots, \varrho_2$ represent the premise variables and the IT2 fuzzy sets, respectively. Define

$$\varpi_j(\mathbf{x}(t)) = [\underline{\varpi}_j(\mathbf{x}(t)), \overline{\varpi}_j(\mathbf{x}(t))], j = 1, 2, \ldots, \varrho_2 \tag{7}$$

as the jth rule's firing strength where

$$\underline{\varpi}_j(\mathbf{x}(t)) = \prod_{\tilde{\sigma}=1}^{\tilde{\Phi}} \underline{\mu}_{\tilde{M}_{\tilde{\sigma}}^j}(g_{\tilde{\sigma}}(\mathbf{x}(t)))$$

$$\overline{\varpi}_j(\mathbf{x}(t)) = \prod_{\tilde{\sigma}=1}^{\tilde{\Phi}} \overline{\mu}_{\tilde{M}_{\tilde{\sigma}}^j}(g_{\tilde{\sigma}}(\mathbf{x}(t)))$$

with $0 \le \underline{\mu}_{M_{\tilde{\sigma}}^j}(g_{\tilde{\sigma}}(\mathbf{x}(t))) \le \overline{\mu}_{M_{\tilde{\sigma}}^j}(g_{\tilde{\sigma}}(\mathbf{x}(t))) \le 1$. Hence, it follows that $0 \le \underline{\varpi}_j(\mathbf{x}(t)) \le \overline{\varpi}_j(\mathbf{x}(t)) \le 1, \forall j$.

Thus, the asynchronous IT2 fuzzy controller in mode ι can be inferred by

$$\mathbf{u}(t) = \sum_{j=1}^{\varrho_2} \varpi_j(\mathbf{x}(t))K_j^\iota(\mathbf{x}(t))\hat{\mathbf{x}}(\mathbf{x}(t)). \tag{8}$$

Considering the property of MFs, i.e.,

$$\sum_i^{\varrho_1} \omega_i(\mathbf{x}) = \sum_j^{\varrho_2} \varpi_j(\mathbf{x}) = \sum_i^{\varrho_1}\sum_j^{\varrho_2} \omega_i(\mathbf{x})\varpi_j(\mathbf{x}) = 1$$

it can be obtained that the compact form for the closed-loop system can be written by

$$\mathcal{G}_{cl} \begin{cases} \dot{\mathbf{x}} = \sum_{i=1}^{\varrho_1}\sum_{j=1}^{\varrho_2} \hbar_{ij}(\mathbf{x})\{\mathcal{A}_{ij}^{p\iota}(\mathbf{x})\hat{\mathbf{x}}(\mathbf{x}) + \mathbf{D}_i^p(\mathbf{x})\mathbf{w}(\mathbf{x})\} \\ \mathbf{z} = \mathbf{C}^p\hat{\mathbf{x}}(\mathbf{x}) \end{cases} \tag{9}$$

where $\mathcal{A}_{ij}^{p\iota}(\mathbf{x}) = \mathbf{A}_i^p(\mathbf{x}) + \mathbf{B}_i^p(\mathbf{x})K_j^\iota(\mathbf{x})$ and $\hbar_{ij}(\mathbf{x}) = \omega_i(\mathbf{x})\varpi_j(\mathbf{x})$. Considering the uncertainties in $\omega_i(\mathbf{x})$ and $\varpi_j(\mathbf{x})$, $\hbar_{ij}(\mathbf{x})$ cannot be used directly in stability analysis and synthesis. Hence, in order to facilitating the controller design scheme, we reconstruct $\hbar_{ij}(\mathbf{x})$ by following the discussions in [33] in the form:

$$\hbar_{ij}(\mathbf{x}) = \underline{\tau}_{ij}(\mathbf{x})\underline{\hbar}_{ij}(\mathbf{x}) + \overline{\tau}_{ij}(\mathbf{x})\overline{\hbar}_{ij}(\mathbf{x}) \tag{10}$$

where $0 \le \underline{\tau}_{ij}(\mathbf{x}) \le \overline{\tau}_{ij}(\mathbf{x}) \le 1$ are nonlinear weighting functions satisfying $\underline{\tau}_{ij}(\mathbf{x}) + \overline{\tau}_{ij}(\mathbf{x}) = 1$. $\underline{\hbar}_{ij}(\mathbf{x})$ and $\overline{\hbar}_{ij}(\mathbf{x})$ can be regarded as the lower and upper MFs. The following proposition is essential for them to be taken into account in the stability analysis.

Proposition 1 (see[28]): Denote the whole state space of interest for $\mathbf{x} = [x_1, x_2, \ldots, x_n]^T$ as Δ. Partition Δ into δ substate space denoted by $\Delta_\rho, \rho = 1, 2, \ldots, \delta$, which satisfies $\bigcup_{\rho=1}^{\delta} \Delta_k = \Delta$. Then, the nonlinear functions $\underline{\hbar}_{ij}(\mathbf{x})$ and $\overline{\hbar}_{ij}(\mathbf{x})$ can be expressed as

$$\begin{cases} \underline{\hbar}_{ij}(\mathbf{x}) = \sum_{\rho=1}^{\delta}\sum_{c_1=1}^{2}\sum_{c_2=1}^{2}\cdots\sum_{c_n=1}^{2}\prod_{r=1}^{n}\varsigma_{ri_r\rho}(x_r)\underline{\eta}_{ijc_1c_2\ldots c_n\rho} \\ \overline{\hbar}_{ij}(\mathbf{x}) = \sum_{\rho=1}^{\delta}\sum_{c_1=1}^{2}\sum_{c_2=1}^{2}\cdots\sum_{c_n=1}^{2}\prod_{r=1}^{n}\varsigma_{ri_r\rho}(x_r)\overline{\eta}_{ijc_1c_2\ldots c_n\rho} \end{cases} \tag{11}$$

where $\varsigma_{ri_r\rho}(x_r)$ are functions satisfying $\sum_{i_r=1}^{2}\varsigma_{ri_r\rho}(x_r) = 1$ with $0 \le \varsigma_{ri_r\rho}(x_r) \le 1$ for all r, ρ, and $\mathbf{x} \in \Delta_\rho$, otherwise, $\varsigma_{ri_r\rho}(x_r) = 0$. Thus, it indicates $\sum_{\rho=1}^{\delta}\sum_{c_1=1}^{2}\sum_{c_2=1}^{2}\cdots\sum_{c_n=1}^{2}\prod_{r=1}^{n}\varsigma_{ri_r\rho}(x_r) = 1$. The parameters $\underline{\eta}_{ijc_1c_2}$ $\ldots c_n\rho$ and $\overline{\eta}_{ijc_1c_2\ldots c_n\rho}$ are predefined constant scalars related to substate space satisfying $0 \le \underline{\eta}_{ijc_1c_2\ldots c_n\rho} \le \overline{\eta}_{ijc_1c_2\ldots c_n\rho} \le 1$.

Definition 1 (Infinitesimal operator [34]): The infinitesimal operator $\Im$ is defined for a Lyapunov functional candidate V as

$$\Im V = \lim_{\Delta\to 0}\frac{1}{\Delta}\left(\mathbb{E}\{\hat{V}(t+\Delta|\mathbf{x}(t), \mathbf{H}(\mathbf{t}))\} - V(\mathbf{x}(t), \mathbf{H}(t))\right) \tag{12}$$

where

$$\hat{V}(t+\Delta|\mathbf{x}(t), \mathbf{H}(\mathbf{t})) = V(\mathbf{x}(t+\Delta), \mathbf{H}(\mathbf{t}+\mathbf{\Delta})|\mathbf{x}(t), \mathbf{H}(\mathbf{t})).$$

Definition 2: The closed-loop system $\mathcal{G}_{cl}$ with $\mathbf{w}(t) \equiv 0$ is said to be stochastically stable, if

$$\int_{t=0}^{\infty}\mathbb{E}\{\|\mathbf{x}(t)\|^2|\mathbf{H}_0 = (p_0, \iota_0)\} < \infty \quad \forall \mathbf{x}_0 \tag{13}$$

holds for and initial condition $(\mathbf{x}_0, \mathbf{H}_0)$.

Definition 3: If closed-loop system $\mathcal{G}_{cl}$ is stochastically stable and the $\mathcal{H}_\infty$ norm satisfies $\|\mathcal{G}_{cl}\|_\infty < \gamma$, i.e.,

$$\int_0^{\infty}\mathbf{z}^T(t)\mathbf{z}(t)\mathrm{d}t < \int_0^{\infty}\gamma^2\mathbf{w}^T(t)\mathbf{w}(t)\mathrm{d}t \tag{14}$$

where $\mathbf{w}(t) \in \mathcal{L}_2[0, +\infty)$, system $\mathcal{G}_{cl}$ is said to be stochastically stable with a predefined $\mathcal{H}_\infty$ performance attenuation level index γ under the given controller (8).

III. Main Results

This section starts with the stability and $\mathcal{H}_\infty$ performance analysis for the considered system and then the asynchronous IT2 polynomial fuzzy controller synthesis is performed. The design scheme is given in terms of solving SOS conditions.

A. Stability and $\mathcal{H}_\infty$ Performance Analysis

For closed-loop system $\mathcal{G}_{cl}$, we define $\mathbf{T}(\mathbf{x}) = \frac{\partial \hat{\mathbf{x}}}{\partial \mathbf{x}}$. Then, $\dot{\hat{\mathbf{x}}}(\mathbf{x}) = \mathbf{T}(\mathbf{x})\dot{\mathbf{x}}$. Hence, we have

$$\dot{\hat{\mathbf{x}}}(\mathbf{x}) = \sum_{i=1}^{k_1}\sum_{j=1}^{k_2} \hbar_{ij}(\mathbf{x})\{\tilde{\mathcal{A}}_{ij}^{p\iota}(\mathbf{x})\hat{\mathbf{x}}(\mathbf{x}) + \tilde{\mathbf{D}}_i^p(\mathbf{x})\mathbf{w}(t)\} \quad (15)$$

where $\tilde{\mathcal{A}}_{ij}^{p\iota}(\mathbf{x}) = \tilde{\mathbf{A}}_i^p(\mathbf{x}) + \tilde{\mathbf{B}}_i^p(\mathbf{x})K_j^\iota(\mathbf{x})$ with $\tilde{\mathbf{A}}_i^p(\mathbf{x}) = \mathbf{T}(\mathbf{x})\mathbf{A}_i^p(\mathbf{x}), \tilde{\mathbf{B}}_i^p(\mathbf{x}) = \mathbf{T}(\mathbf{x})\mathbf{B}_i^p(\mathbf{x})$ and $\tilde{\mathbf{D}}_i^p(\mathbf{x}) = \mathbf{T}(\mathbf{x})\mathbf{D}_i^p(\mathbf{x})$.

The compact form can be written as follows for lightening notations:

$$\begin{cases}\dot{\hat{\mathbf{x}}}(\mathbf{x}) = \sum_{i=1}^{\varrho_1}\sum_{j=1}^{\varrho_2} \hbar_{ij}\Gamma_{ij}^{p\iota}\mathbf{v} \\ \mathbf{z} = \Sigma^p\mathbf{v}\end{cases} \quad (16)$$

where $\mathbf{v} = \begin{bmatrix}\hat{\mathbf{x}}(\mathbf{x})^T & \mathbf{w}^T(t)\end{bmatrix}^T$, $\Sigma^p = \begin{bmatrix}\mathbf{C}^p & 0\end{bmatrix}$ and

$$\Gamma_{ij}^{p\iota} = \begin{bmatrix}\tilde{\mathcal{A}}_{ij}^{p\iota}(\mathbf{x}) & \tilde{\mathbf{D}}_i^p(\mathbf{x})\end{bmatrix}.$$

Theorem 1: Consider a nonlinear MJS described in the form of IT2 polynomial fuzzy model (5) with a well-designed controller (8). Given scalars $\gamma > 0$, the closed-loop system (9) is stochastically stable with a guaranteed $\mathcal{H}_\infty$ performance index γ, if there exist polynomial matrices $Q(\mathbf{x}), W_{ij}(\mathbf{x}) = W_{ij}^T(\mathbf{x}), \mathbf{G}_\rho(\mathbf{x}) = \mathbf{G}_\rho^T(\mathbf{x})$ and symmetric matrices $\mathbf{X}_{p\iota}$ such that the following SOS-based conditions hold for rules $i = 1, 2, \ldots, \varrho_1, j = 1, 2, \ldots, \varrho_2$ and all $p \in \mathcal{N}, \iota \in \mathcal{S}$

$$\nu_1^T(\mathbf{X}_{p\iota} - \zeta_1 I)\nu_1 \;\text{ is SOS} \quad (17)$$

$$\nu_2^T(W_{ij}(\mathbf{x}) - \zeta_2(\mathbf{x})I)\nu_2 \;\text{ is SOS} \quad (18)$$

$$\nu_2^T(\mathbf{G}_\rho(\mathbf{x}) - \zeta_3(\mathbf{x})I)\nu_2 \;\text{ is SOS} \quad (19)$$

$$\nu_2^T(\tilde{\Omega}_{ij}^{p\iota} + W_{ij}(\mathbf{x}) + Q(\mathbf{x}) - \zeta_4(\mathbf{x})I)\nu_2 \;\text{ is SOS} \quad (20)$$

$$-\nu_2^T\Bigg\{\sum_{i=1}^{\varrho_1}\sum_{j=1}^{\varrho_2}\Big(\underline{\eta}_{ijc_1c_2\ldots c_n\rho}\tilde{\Omega}_{ij}^{p\iota} - \big(\underline{\eta}_{ijc_1c_2\ldots c_n\rho} - \overline{\eta}_{ijc_1c_2\ldots c_n\rho}\big) \cdot W_{ij}(\mathbf{x}) + \overline{\eta}_{ijc_1c_2\ldots c_n\rho}Q(\mathbf{x})\Big) + (\mathbf{x} - \underline{\mathbf{x}}_\rho)^T\mathbf{\Lambda}(\overline{\mathbf{x}}_\rho - \mathbf{x})\mathbf{G}_\rho(\mathbf{x}) - Q(\mathbf{x}) + \zeta_5(\mathbf{x})I\Bigg\}\nu_2 \;\text{ is SOS},\; \forall c_1, c_2, \ldots, c_n, \rho \quad (21)$$

where ζ_1 is a given positive scalar and $\zeta_d(\mathbf{x}) > 0, d = 2, 3, 4, 5$ are predefined positive polynomials. $\underline{\eta}_{ijc_1c_2\ldots c_n\rho}, \overline{\eta}_{ijc_1c_2\ldots c_n\rho}$ are known parameters defined in Proposition 1. The arbitrary vectors ν_1, ν_2 are supposed to be independent of $\mathbf{x}$. Matrix $\tilde{\Omega}_{ij}^{p\iota}$ is defined as

$$\tilde{\Omega}_{ij}^{p\iota} = \begin{bmatrix}\Psi + \sum_{(q,\kappa)\neq(p,\iota)}\vartheta_{(p,\iota)(q,\kappa)}\mathbf{X}_{q\kappa} + \mathbf{C}^{pT}\mathbf{C}^p & \mathbf{X}_{p\iota}\tilde{\mathbf{D}}_i^p(\mathbf{x}) \\ \tilde{\mathbf{D}}_i^{pT}(\mathbf{x})\mathbf{X}_{p\iota} & -\gamma^2 I\end{bmatrix}$$

with

$$\Psi = \mathbf{He}\{\mathbf{X}_{p\iota}\tilde{\mathcal{A}}_{ij}^{p\iota}(\mathbf{x})\} + \vartheta_{(p,\iota)(p,\iota)}\mathbf{X}_{p\iota}.$$

Proof: First, we construct the following quadratic value function:

$$V(\hat{\mathbf{x}}(\mathbf{x}), p, \iota) = \hat{\mathbf{x}}(\mathbf{x})^T\mathbf{X}_{p\iota}\hat{\mathbf{x}}(\mathbf{x})$$

with $\mathbf{X}_{p\iota} = \mathbf{X}_{p\iota}^T > 0$. Along the trajectories of system (9), we can obtain that the weak infinitesimal operator of V is

$$\begin{aligned}\Im V &= \dot{\hat{\mathbf{x}}}(\mathbf{x})^T\mathbf{X}_{p\iota}\hat{\mathbf{x}}(\mathbf{x}) + \hat{\mathbf{x}}(\mathbf{x})^T\mathbf{X}_{p\iota}\dot{\hat{\mathbf{x}}}(\mathbf{x}) \\ &\quad + \hat{\mathbf{x}}(\mathbf{x})^T\left(\sum_{(q,\kappa)\in\mathcal{V}}\vartheta_{(p,\iota)(q,\kappa)}\mathbf{X}_{q\kappa}\right)\hat{\mathbf{x}}(\mathbf{x}) \\ &= \sum_{i=1}^{\varrho_1}\sum_{j=1}^{\varrho_2}\hbar_{ij}(\mathbf{x})(\mathbf{v}^T\Gamma_{ij}^{p\iota T}\mathbf{X}_{p\iota}\hat{\mathbf{x}}(\mathbf{x}) + \hat{\mathbf{x}}(\mathbf{x})^T\mathbf{X}_{p\iota}\Gamma_{ij}^{p\iota}\mathbf{v}) \\ &\quad + \hat{\mathbf{x}}(\mathbf{x})^T\left(\sum_{(q,\kappa)\in\mathcal{V}}\vartheta_{(p,\iota)(q,\kappa)}\mathbf{X}_{q\kappa}\right)\hat{\mathbf{x}}(\mathbf{x}) \\ &= \sum_{i=1}^{\varrho_1}\sum_{j=1}^{\varrho_2}\hbar_{ij}(\mathbf{x})\mathbf{v}^T\Omega_{ij}^{p\iota}\mathbf{v}\end{aligned} \quad (22)$$

where

$$\Omega_{ij}^{p\iota} = \begin{bmatrix}\Psi + \sum_{(q,\kappa)\neq(p,\iota)}\vartheta_{(p,\iota)(q,\kappa)}\mathbf{X}_{q\kappa} & \mathbf{X}_{p\iota}\tilde{\mathbf{D}}_i^p(\mathbf{x}) \\ \tilde{\mathbf{D}}_i^{pT}(\mathbf{x})\mathbf{X}_{p\iota} & 0\end{bmatrix}.$$

When $\mathbf{w}(t) \equiv 0$, it can be seen that $\Im V < 0$, i.e., the stability of (9) can be guaranteed by the condition of

$$\sum_{i=1}^{\varrho_1}\sum_{j=1}^{\varrho_2}\hbar_{ij}(\mathbf{x})\left(\Psi + \sum_{(q,\kappa)\neq(p,\iota)}\vartheta_{(p,\iota)(q,\kappa)}\mathbf{X}_{q\kappa}\right) < 0. \quad (23)$$

Furthermore, we consider the $\mathcal{H}_\infty$ performance index under zero initial condition

$$\mathbf{J} = \int_0^\infty \mathbf{z}^T\mathbf{z}\mathrm{d}t - \int_0^\infty \gamma^2\mathbf{w}^T\mathbf{w}\mathrm{d}t. \quad (24)$$

Suppose the dynamics can be stabilized by the controller, then

$$\begin{aligned}\mathbf{J} &\leq \int_0^\infty(\Im V + \mathbf{z}^T\mathbf{z} - \gamma^2\mathbf{w}^T\mathbf{w})\mathrm{d}t \\ &= \int_0^\infty\left(\sum_{i=1}^{\varrho_1}\sum_{j=1}^{\varrho_2}\hbar_{ij}(\mathbf{x})\mathbf{v}^T\tilde{\Omega}_{ij}^{p\iota}\mathbf{v}\right)\mathrm{d}t\end{aligned} \quad (25)$$

where

$$\tilde{\Omega}_{ij}^{p\iota} = \begin{bmatrix} \Psi+\sum_{(q,\kappa)\neq(p,\iota)}\vartheta_{(p,\iota)(q,\kappa)}\mathbf{X}_{q\kappa}+\mathbf{C}^{pT}\mathbf{C}^{p} & \mathbf{X}_{p\iota}\tilde{\mathbf{D}}_i^p(\mathbf{x}) \\ \tilde{\mathbf{D}}_i^{pT}(\mathbf{x})\mathbf{X}_{p\iota} & -\gamma^2 I \end{bmatrix}.$$

$\mathbf{J}<0$ can be ensured by

$$\sum_{i=1}^{\varrho_1}\sum_{j=1}^{\varrho_2}\hbar_{ij}(\mathbf{x})\tilde{\Omega}_{ij}^{p\iota}<0. \tag{26}$$

To facilitate the MFD approach, slack matrices $W_{ij}(\mathbf{x})=W_{ij}^T(\mathbf{x})\geq 0$ and $Q(\mathbf{x})=Q^T(\mathbf{x})$ are introduced. They satisfy

$$\begin{cases}\left(\sum_{i=1}^{\varrho_1}\sum_{j=1}^{\varrho_2}(\underline{\tau}_{ij}(\mathbf{x})\underline{\hbar}_{ij}(\mathbf{x})+\overline{\tau}_{ij}(\mathbf{x})\overline{\hbar}_{ij}(\mathbf{x}))-1\right)Q(\mathbf{x})=0\\ -\sum_{i=1}^{\varrho_1}\sum_{j=1}^{\varrho_2}(1-\underline{\tau}_{ij}(\mathbf{x}))(\underline{\hbar}_{ij}(\mathbf{x})-\overline{\hbar}_{ij}(\mathbf{x}))W_{ij}(\mathbf{x})\geq 0.\end{cases} \tag{27}$$

Now incorporate the abovementioned expressions into (26). Recall the fact $\underline{\tau}_{ij}(\mathbf{x})+\overline{\tau}_{ij}(\mathbf{x})=1$ and by grouping the terms with $\underline{\tau}_{ij}(\mathbf{x})$ we can obtain

$$\begin{aligned}&\sum_{i=1}^{\varrho_1}\sum_{j=1}^{\varrho_2}\hbar_{ij}(\mathbf{x})\tilde{\Omega}_{ij}^{p\iota}\\ &\leq\sum_{i=1}^{\varrho_1}\sum_{j=1}^{\varrho_2}\underline{\tau}_{ij}(\mathbf{x})(\underline{\hbar}_{ij}(\mathbf{x})-\overline{\hbar}_{ij}(\mathbf{x}))(\tilde{\Omega}_{ij}^{p\iota}+W_{ij}(\mathbf{x})+Q(\mathbf{x}))\\ &\quad+\sum_{i=1}^{\varrho_1}\sum_{j=1}^{\varrho_2}(\underline{\hbar}_{ij}(\mathbf{x})\tilde{\Omega}_{ij}^{p\iota}-(\underline{\hbar}_{ij}(\mathbf{x})-\overline{\hbar}_{ij}(\mathbf{x}))W_{ij}(\mathbf{x})\\ &\quad+\overline{\hbar}_{ij}(\mathbf{x})Q(\mathbf{x}))-Q(\mathbf{x})\end{aligned} \tag{28}$$

which shows that $\mathbf{J}<0$ holds if

$$\tilde{\Omega}_{ij}^{p\iota}+W_{ij}(\mathbf{x})+Q(\mathbf{x})>0 \tag{29}$$

$$\begin{aligned}&\sum_{i=1}^{\varrho_1}\sum_{j=1}^{\varrho_2}(\underline{\hbar}_{ij}(\mathbf{x})\tilde{\Omega}_{ij}^{p\iota}-(\underline{\hbar}_{ij}(\mathbf{x})-\overline{\hbar}_{ij}(\mathbf{x}))W_{ij}(\mathbf{x})\\ &\qquad+\overline{\hbar}_{ij}(\mathbf{x})Q(\mathbf{x}))-Q(\mathbf{x})<0.\end{aligned} \tag{30}$$

Consequently, by substituting (11) into (30), we have the following equivalent inequality:

$$\begin{aligned}&\sum_{\rho=1}^{\delta}\sum_{c_1=1}^{2}\sum_{c_2=1}^{2}\cdots\sum_{c_n=1}^{2}\prod_{r=1}^{n}\varsigma_{ri_r\rho}(x_r)\sum_{i=1}^{\varrho_1}\sum_{j=1}^{\varrho_2}\Big(\underline{\eta}_{ijc_1c_2\ldots c_n\rho}\tilde{\Omega}_{ij}^{p\iota}\\ &-(\underline{\eta}_{ijc_1c_2\ldots c_n\rho}-\overline{\eta}_{ijc_1c_2\ldots c_n\rho})W_{ij}(\mathbf{x})+\overline{\eta}_{ijc_1c_2\ldots c_n\rho}Q(\mathbf{x})\Big)\\ &-Q(\mathbf{x})<0\end{aligned}$$

which means that

$$\begin{aligned}&\sum_{i=1}^{\varrho_1}\sum_{j=1}^{\varrho_2}\Big(\underline{\eta}_{ijc_1c_2\ldots c_n\rho}\tilde{\Omega}_{ij}^{p\iota}-(\underline{\eta}_{ijc_1c_2\ldots c_n\rho}-\overline{\eta}_{ijc_1c_2\ldots c_n\rho})W_{ij}(\mathbf{x})\\ &\qquad+\overline{\eta}_{ijc_1c_2\ldots c_n\rho}Q(\mathbf{x})\Big)-Q(\mathbf{x})<0.\end{aligned} \tag{31}$$

Note that the abovementioned inequality should be validated in each subspace Δ_ρ, which is element wisely bounded by $\underline{\mathbf{x}}_\rho$ and $\overline{\mathbf{x}}_\rho$. Define a slack polynomial matrix $\mathbf{G}_\rho(\mathbf{x})=\mathbf{G}_\rho^T(\mathbf{x})\geq 0, \rho=1,2\ldots,\delta$. It naturally satisfies that

$$(\mathbf{x}-\underline{\mathbf{x}}_\rho)^T\mathbf{\Lambda}(\overline{\mathbf{x}}_\rho-\mathbf{x})\mathbf{G}_\rho(\mathbf{x})\geq 0 \quad \forall\rho,\mathbf{x}\in\Delta_\rho \tag{32}$$

where $\mathbf{\Lambda}=\{\Lambda_1,\Lambda_2,\ldots,\Lambda_n\}$ is a diagonal matrix with Λ_i being either 1 or 0, $i=1,2,\ldots,n$. The rule is that when the state x_i is involved in the analysis, $\Lambda_i=1$; otherwise, $\Lambda_i=0$.

By combining (31) and (32), we have

$$\begin{aligned}&\sum_{i=1}^{\varrho_1}\sum_{j=1}^{\varrho_2}\Big(\underline{\eta}_{iji_1i_2\ldots i_n\rho}\tilde{\Omega}_{ij}^{p\iota}-\left(\underline{\eta}_{iji_1i_2\ldots i_n\rho}-\overline{\eta}_{iji_1i_2\ldots i_n\rho}\right)W_{ij}(\mathbf{x})\\ &\qquad+\overline{\eta}_{iji_1i_2\ldots i_n\rho}Q(\mathbf{x})\Big)-Q(\mathbf{x})\\ &\qquad+(\mathbf{x}-\underline{\mathbf{x}}_\rho)^T\mathbf{\Lambda}(\overline{\mathbf{x}}_\rho-\mathbf{x})\mathbf{G}_\rho(\mathbf{x})<0.\end{aligned} \tag{33}$$

Therefore, it can be concluded that the feasibility of (29) and (33) guarantees (26), which implies (23) and $\mathbf{J}<0$ hold. By Definitions 1 and 2, the proof is completed. ∎

Remark 2: In many works concerning the analysis for fuzzy MJS, sufficient conditions, say $\tilde{\Omega}_{ij}^{p\iota}\leq 0$ for $1\leq i,j\leq s$ and $(p,\iota)\in\mathcal{V}$ is enough guaranteeing the stability of the system. However, the conservatism is apparent. The MFs are not utilized into the final conditions, which means that the results are obtained for systems with the same subsystem structure but not necessarily the same MFs. In the fuzzy modeling process for a nonlinear system, the MFs preserve its nonlinear traits, thus, will specify the results for the considered system.

Remark 3: It can be noted that it is impossible to validate condition (26) for every point of $\mathbf{x}(t)$ due to the existence of MFs. By introducing the piecewise linear function approximation method, condition (26) is transformed into (33) and can be checked by validating finite number of points.

Remark 4: The introducing of slack matrices $W_{ij}(\mathbf{x})$ and $Q(\mathbf{x})$ helps facilitating the MFD approach, while as the number of variables increase the computation demand becomes severe since all elements of these matrices are to be determined. It can be reduced by setting them to be diagonal matrices. However, the conservatism may be introduced.

B. IT2 Polynomial Fuzzy Asynchronous Controller Synthesis

Theorem 2: Consider a nonlinear MJS described in the form of IT2 polynomial fuzzy model (5) with a well-designed controller (8). Given scalars $\gamma>0,\xi_j^\iota>0$, the closed-loop system (9) is stochastically stable with a guaranteed $\mathcal{H}_\infty$ performance index γ, if there exist matrices $\mathbf{R}_{p\iota}=\mathbf{R}_{p\iota}^T,\mathbf{Y}_{p\iota},U_\iota$ and polynomial matrices $\mathbf{L}_j^\iota(\mathbf{x}),Q(\mathbf{x}),W_{ij}(\mathbf{x})=W_{ij}^T(\mathbf{x}),\mathbf{G}_\rho(\mathbf{x})=\mathbf{G}_\rho^T(\mathbf{x})$such that the following SOS-based conditions hold for rules $i=1,2,\ldots,\varrho_1,j=1,2,\ldots,\varrho_2$ and all $p\in\mathcal{N},\iota\in\mathcal{S}$

$$\nu_1^T(\mathbf{R}_{p\iota}-\zeta_1 I)\nu_1 \ \text{ is SOS} \tag{34}$$

$$\nu_2^T(W_{ij}(\mathbf{x})-\zeta_2(\mathbf{x})I)\nu_2 \ \text{ is SOS} \tag{35}$$

$$\nu_2^T(\mathbf{G}_\rho(\mathbf{x})-\zeta_3(\mathbf{x})I)\nu_2 \ \text{ is SOS} \tag{36}$$

$$\nu_2^T(\tilde{\mathbf{\Omega}}_{ij}^{p\iota}+W_{ij}(\mathbf{x})+Q(\mathbf{x})-\zeta_4(\mathbf{x})I)\nu_2 \ \text{ is SOS} \tag{37}$$

$$-\nu_2^T\left\{\sum_{i=1}^{\varrho_1}\sum_{j=1}^{\varrho_2}\left(\underline{\eta}_{ijc_1c_2\ldots c_n\rho}\tilde{\mathbf{\Omega}}_{ij}^{p\iota}-\left(\underline{\eta}_{ijc_1c_2\ldots c_n\rho}-\overline{\eta}_{ijc_1c_2\ldots c_n\rho}\right)\right.\right.$$
$$\left.\cdot W_{ij}(\mathbf{x})+\overline{\eta}_{ijc_1c_2\ldots c_n\rho}Q(\mathbf{x})\right)+(\mathbf{x}-\underline{\mathbf{x}}_\rho)^T\mathbf{\Lambda}(\overline{\mathbf{x}}_\rho-\mathbf{x})\mathbf{G}_\rho(\mathbf{x})$$
$$\left.-\,Q(\mathbf{x})+\zeta_5(\mathbf{x})I\right\}\nu_2\ \text{ is SOS }\quad\forall c_1,c_2,\ldots,c_n,\rho \tag{38}$$

where $\underline{\eta}_{ijc_1c_2\ldots c_n\rho},\overline{\eta}_{ijc_1c_2\ldots c_n\rho}$ are parameters defined in Proposition 1. And $\tilde{\mathbf{\Omega}}_{ij}^{p\iota}$

$$=\begin{bmatrix}\Upsilon_{ij}^{p\iota} & \tilde{\mathbf{B}}_i^p(\mathbf{x})\mathbf{L}_j^\iota(\mathbf{x}) & 0 & \tilde{\mathbf{D}}_i^p(\mathbf{x}) & \mathbf{R}_{p\iota}\mathbf{C}^{pT} & \tilde{\mathbb{X}}\\ * & -\mathbf{He}(U_\iota) & \xi_j^\iota\mathbf{R}_{p\iota}-U_\iota & 0 & 0 & 0\\ * & * & -\mathbf{Y}_{p\iota} & 0 & 0 & 0\\ * & * & * & -\gamma^2 I & 0 & 0\\ * & * & * & * & -I & 0\\ * & * & * & * & * & -\tilde{\mathbf{X}}\end{bmatrix}$$

with

$$\Upsilon_{ij}^{p\iota}=\mathbf{He}\left\{\tilde{\mathbf{A}}_i^p(\mathbf{x})\mathbf{R}_{p\iota}+\tilde{\mathbf{B}}_i^p(\mathbf{x})\mathbf{L}_j^\iota(\mathbf{x})\right\}+\vartheta_{(p,\iota)(p,\iota)}\mathbf{R}_{p\iota}+\mathbf{Y}_{p\iota}$$
$$\tilde{\mathbb{X}}=\mathbf{R}_{p\iota}\left[\sqrt{\vartheta_{(p,\iota)\phi_{(p,\iota)}(1)}}I,\ldots,\sqrt{\vartheta_{(p,\iota)\phi_{(p,\iota)}(\psi_{(p,\iota)})}}I\right]$$
$$\tilde{\mathbf{X}}=\ \text{diag}\left\{\mathbf{R}_{\phi_{(p,\iota)}(1)},\ldots,\mathbf{R}_{\phi_{(p,\iota)}(\psi_{(p,\iota)})}\right\}$$
$$\mathcal{V}_{(p,\iota)}\triangleq\left\{(q,\kappa)\in\mathcal{V};(q,\kappa)\neq(p,\iota)\ \text{and}\ \vartheta_{(p,\iota)(q,\kappa)}\neq 0\right\}$$
$$\triangleq\left\{\phi_{(p,\iota)}(1),\ldots,\phi_{(p,\iota)}(\psi_{(p,\iota)})\right.$$
$$\left.\phi_{(p,\iota)}(\tilde{k})\in\mathcal{V},\tilde{k}=1,\ldots,\psi_{(p,\iota)}\right\}.$$

Then, the IT2 polynomial fuzzy asynchronous controller gain can be obtained by

$$K_j^\iota(\mathbf{x})=\xi_j^\iota\mathbf{L}_j^\iota(\mathbf{x})U_\iota^{-1}. \tag{39}$$

Proof: From the discussions in Theorem 1, we know that the feasibility of (26) sufficiently concludes the stability with a predefined $\mathcal{H}_\infty$ performance for system (9).

Now following the similar line processed in Theorem 1, we have that (37) and (38) ensures

$$\sum_{i=1}^{\varrho_1}\sum_{j=1}^{\varrho_2}\hbar_{ij}(\mathbf{x})\tilde{\mathbf{\Omega}}_{ij}^{p\iota}<0. \tag{40}$$

Recall the fact that $\sum_{i=1}^{\varrho_1}\sum_{j=1}^{\varrho_2}\hbar_{ij}(\mathbf{x})=1$ and further by schur complement, we have

$$\sum_{i=1}^{\varrho_1}\sum_{j=1}^{\varrho_2}\hbar_{ij}(\mathbf{x})\begin{bmatrix}\mathbf{\Upsilon}_{ij}^{p\iota} & \tilde{\mathbf{D}}_i^p(\mathbf{x}) & \tilde{\mathbf{B}}_i^p(\mathbf{x})\mathbf{L}_j^\iota(\mathbf{x})\\ * & -\gamma^2 I & 0\\ * & * & \mathbf{U}_{p\iota}\end{bmatrix}<0 \tag{41}$$

where $\mathbb{X}=\mathbf{R}_{p\iota}\left(\sum_{(q,\kappa)\neq(p,\iota)}\vartheta_{(p,\iota)(q,\kappa)}\mathbf{R}_{q\kappa}^{-1}\right)\mathbf{R}_{p\iota}$ and

$$\mathbf{\Upsilon}_{ij}^{p\iota}=\Upsilon_{ij}^{p\iota}+\mathbb{X}+\mathbf{R}_{p\iota}\mathbf{C}^{pT}\mathbf{C}^p\mathbf{R}_{p\iota}$$
$$\mathbf{U}_{p\iota}=\ -\mathbf{He}(U_\iota)+(\xi_j^\iota\mathbf{R}_{p\iota}-U_\iota)\mathbf{Y}_{p\iota}^{-1}(\xi_j^\iota\mathbf{R}_{p\iota}-U_\iota)^T.$$

Since for any matrices $A,P>0$ with appropriate dimensions, the following always holds:

$$A^TP^{-1}A\geq A^T+A-P.$$

Hence, we can obtain that

$$\mathbf{U}_{p\iota}\geq -U_\iota^T\left((\xi_j^\iota\mathbf{R}_{p\iota}-U_\iota)\mathbf{Y}_{p\iota}^{-1}(\xi_j^\iota\mathbf{R}_{p\iota}-U_\iota)^T\right)^{-1}U_\iota\triangleq\tilde{\mathbf{U}}_{p\iota}$$

which shows that (41) ensures

$$\sum_{i=1}^{\varrho_1}\sum_{j=1}^{\varrho_2}\hbar_{ij}(\mathbf{x})\begin{bmatrix}\mathbf{\Upsilon}_{ij}^{p\iota} & \tilde{\mathbf{D}}_i^p(\mathbf{x}) & \tilde{\mathbf{B}}_i^p(\mathbf{x})\mathbf{L}_j^\iota(\mathbf{x})\\ * & -\gamma^2 I & 0\\ * & * & \tilde{\mathbf{U}}_{p\iota}\end{bmatrix}<0. \tag{42}$$

By matrix manipulating, it follows that

$$\mathbf{\Upsilon}_{ij}^{p\iota}+\tilde{\mathbf{B}}_i^p(\mathbf{x})\mathbf{L}_j^\iota(\mathbf{x})(-\tilde{\mathbf{U}}_{p\iota}^{-1})\tilde{\mathbf{B}}_i^p(\mathbf{x})\mathbf{L}_j^\iota(\mathbf{x})$$
$$=\mathbf{\Upsilon}_{ij}^{p\iota}+\tilde{\mathbf{B}}_i^p(\mathbf{x})\mathbf{L}_j^\iota(\mathbf{x})\left(U_\iota^T\left((\xi_j^\iota\mathbf{R}_{p\iota}-U_\iota)\mathbf{Y}_{p\iota}^{-1}\right.\right.$$
$$\left.\left.\cdot(\xi_j^\iota\mathbf{R}_{p\iota}-U_\iota)^T\right)^{-1}U_\iota\right)^{-1}(\tilde{\mathbf{B}}_i^p(\mathbf{x})\mathbf{L}_j^\iota(\mathbf{x}))^T$$
$$=\mathbf{\Upsilon}^{p\iota}+\tilde{\mathbf{B}}_i^p(\mathbf{x})(K_j^\iota(\mathbf{x})\mathbf{R}_{p\iota}-\mathbf{L}_j^\iota)\mathbf{Y}_{p\iota}^{-1}$$
$$+(K_j^\iota(\mathbf{x})\mathbf{R}_{p\iota}-\mathbf{L}_j^\iota)^T\tilde{\mathbf{B}}_i^p(\mathbf{x})^T$$
$$\geq\mathbf{\Upsilon}_{ij}^{p\iota}+\mathbf{He}\{\tilde{\mathbf{B}}_i^p(\mathbf{x})(K_j^\iota(\mathbf{x})\mathbf{R}_{p\iota}-\mathbf{L}_j^\iota)\}-\mathbf{Y}_{p\iota}$$
$$=\mathbf{He}\{(\tilde{\mathbf{A}}_i^p(\mathbf{x})+\tilde{\mathbf{B}}_i^p(\mathbf{x})K_j^\iota(\mathbf{x}))\mathbf{R}_{p\iota}\}+\vartheta_{(p,\iota)(p,\iota)}\mathbf{R}_{p\iota}$$
$$+\mathbf{R}_{p\iota}\left(\sum_{(q,\kappa)\neq(p,\iota)}\vartheta_{(p,\iota)(q,\kappa)}\mathbf{R}_{q\kappa}^{-1}\right)\mathbf{R}_{p\iota}+\mathbf{R}_{p\iota}\mathbf{C}^{pT}\mathbf{C}^p\mathbf{R}_{p\iota}$$
$$=\mathbf{R}_{p\iota}\left(\mathbf{He}\{\mathbf{R}_{p\iota}^{-1}(\tilde{\mathbf{A}}_i^p(\mathbf{x})+\tilde{\mathbf{B}}_i^p(\mathbf{x})K_j^\iota(\mathbf{x}))\}+\vartheta_{(p,\iota)(p,\iota)}\mathbf{R}_{p\iota}^{-1}\right.$$
$$\left.+\sum_{(q,\kappa)\neq(p,\iota)}\vartheta_{(p,\iota)(q,\kappa)}\mathbf{R}_{q\kappa}^{-1}+\mathbf{C}^{pT}\mathbf{C}^p\right)\mathbf{R}_{p\iota}$$
$$=\mathbf{R}_{p\iota}\left(\mathbf{He}\{\mathbf{X}_{p\iota}\tilde{\mathcal{A}}_{ij}^{p\iota}(\mathbf{x})\}+\sum_{(q,\kappa)\in\mathcal{V}}\vartheta_{(p,\iota)(q,\kappa)}\mathbf{X}_{q\kappa}\right.$$
$$\left.+\mathbf{C}^{pT}\mathbf{C}^p\right)\mathbf{R}_{p\iota}$$
$$=\mathbf{R}_{p\iota}\mathbf{\Psi}\mathbf{R}_{p\iota}$$

with $\mathbf{X}_{p\iota}^{-1}\triangleq\mathbf{R}_{p\iota}$ and

$$\mathbf{\Psi}\triangleq\mathbf{He}\{\mathbf{X}_{p\iota}\tilde{\mathcal{A}}_{ij}^{p\iota}(\mathbf{x})\}+\sum_{(q,\kappa)\in\mathcal{V}}\vartheta_{(p,\iota)(q,\kappa)}\mathbf{X}_{q\kappa}+\mathbf{C}^{pT}\mathbf{C}^p.$$

Then, let $\tilde{\mathbf{T}}=\text{diag}\{\mathbf{R}_{p\iota},I\}$ and by Schur complement, (42) guarantees thus that

$$\tilde{\mathbf{T}}\cdot\sum_{i=1}^{\varrho_1}\sum_{j=1}^{\varrho_2}\hbar_{ij}(\mathbf{x})\begin{bmatrix}\mathbf{\Psi} & \mathbf{X}_{p\iota}\tilde{\mathbf{D}}_i^p(\mathbf{x})\\ * & -\gamma^2 I\end{bmatrix}\cdot\tilde{\mathbf{T}}^T<0 \tag{43}$$

i.e.,

$$\sum_{i=1}^{\varrho_1}\sum_{j=1}^{\varrho_2}\hbar_{ij}(\mathbf{x})\tilde{\Omega}_{ij}^{p\iota}<0. \tag{44}$$

Therefore, the proof is completed. ■

Remark 5: The use of joint transition rate $\vartheta_{(p,\iota)(q,\kappa)}$ helps facilitate the derivation of MFD conditions for asynchronous controller synthesis. In this context, it is more straightforward compared with the mode separation strategy adopted in the literature [6] and [21].

Remark 6: Theorem 2 presents solvable SOS-based conditions for obtaining the desired asynchronous controller gains. According to the discussions in Remark 1, all cases of asynchronous degree can be described by different settings of transition rate $\vartheta_{(p,\iota)(q,\kappa)}$. Hence, Theorem 2 provides generalized solution to control issues with mode information mismatching problems.

Remark 7: Regarding the computation complexity of Theorem 2, it has $(2N+1+\varrho_2)S+\varrho_1\varrho_2+\delta+1$ SOS variables and $NS+\varrho_1\varrho_2+(2^n+1)\delta$ SOS-based conditions. When the state space Δ is divide into more substate space, i.e., a larger δ, the piecewise linear function has a stronger approximation, which will lead to a more relaxed result but increase the computation burden as well.

IV. Numerical Examples

Example 1: For illustration purpose, we present a nonlinear system of interest, which has the same form as in (5) with two operation modes. The system matrices with polynomials are arbitrarily chosen as

$$\mathbf{A}_1^1(\mathbf{x}(t))=\begin{bmatrix}-0.20x_1(t)+0.59 & -1.82x_1(t)-7.92\\ 0.01 & -2.85\end{bmatrix}$$

$$\mathbf{A}_2^1(\mathbf{x}(t))=\begin{bmatrix}2.13x_1(t)-0.02 & 0.68x_1(t)-4.64\\ 0.29 & -8.21\end{bmatrix}$$

$$\mathbf{A}_3^1(\mathbf{x}(t))=\begin{bmatrix}0.42x_1(t)+0.73 & 2.22x_1(t)+8.45\\ 0.26 & -15.43\end{bmatrix}$$

$$\mathbf{A}_1^2(\mathbf{x}(t))=\begin{bmatrix}-1.12x_1(t)+5.59 & -1.82x_1(t)-7.29\\ 0.01 & -2.85\end{bmatrix}$$

$$\mathbf{A}_2^2(\mathbf{x}(t))=\begin{bmatrix}2.25x_1(t)-10.02 & 5.72x_1(t)-15.64\\ -10.21 & 8.44\end{bmatrix}$$

$$\mathbf{A}_3^2(\mathbf{x}(t))=\begin{bmatrix}0.50x_1(t)-0.73 & 2.31x_1(t)+8.45\\ 0.26 & 15.43\end{bmatrix}$$

$$\mathbf{B}_1^p(\mathbf{x}(t))=\begin{bmatrix}2.33x_1(t)^2+1.35x_1(t)+1\\ -0.5\end{bmatrix}$$

$$\mathbf{B}_2^p(\mathbf{x}(t))=\begin{bmatrix}2\\ -10.5\end{bmatrix}$$

TABLE I
MFs of the Considered Fuzzy MJS With Uncertainties

$\overline{\omega}_1(\mathbf{x}(t))=1-\frac{1}{1+\exp(-2.5-x_1(t))}$
$\overline{\omega}_3(\mathbf{x}(t))=\frac{1}{1+\exp(2.5-x_1(t))}$
$\underline{\omega}_2(\mathbf{x}(t))=1-\overline{\omega}_1(\mathbf{x}(t))-\overline{\omega}_3(\mathbf{x}(t))$
$\underline{\omega}_1(\mathbf{x}(t))=1-\frac{1}{1+\exp(-3.5-x_1(t))}$
$\underline{\omega}_3(\mathbf{x}(t))=\frac{1}{1+\exp(3.5-x_1(t))}$
$\overline{\omega}_2(\mathbf{x}(t))=1-\underline{\omega}_1(\mathbf{x}(t))-\underline{\omega}_3(\mathbf{x}(t))$

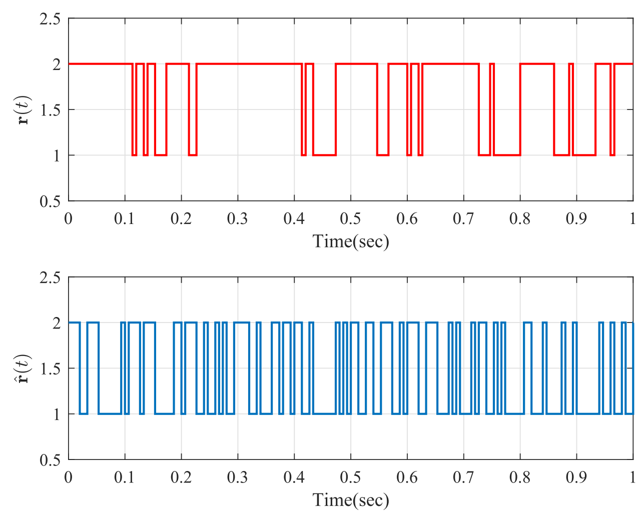

Fig. 1. Possible trajectories of system mode and controller mode.

$$\mathbf{B}_2^p(\mathbf{x}(t))=\begin{bmatrix}3.55x_1(t)^2-0.73x_1(t)+4\\ 0.8\end{bmatrix}$$

with $p\in\{1,2\}$ referring the system modes. The measured output matrix is set to be constant with $\mathbf{C}=\begin{bmatrix}1 & 0\end{bmatrix}$. $\mathbf{D}=\begin{bmatrix}0.5 & 0.8\end{bmatrix}^T$. The IT2 MFs are presented in Table I.

Here, the system operation modes are assumed to have transitions following a Markov process $\{\mathbf{r}(t)\}$, which is characterized by the transition rate matrix

$$\lambda_{pq}=\begin{bmatrix}-2.5 & 2.5\\ 0.6 & -0.6\end{bmatrix}. \tag{45}$$

An asynchronous fuzzy controller in IT2 polynomial form with two fuzzy rules and two modes is employed here to validate our design scheme. Without loss of generality, here the spontaneous jump parameter $\beta_{\iota\kappa}^p$ is set to be equal with value of 1 for any p,ι,κ. And the parameter, which indicates the simultaneous jumps, is set as $\alpha_{q\kappa}^{\iota}=0.3$ for $q=\kappa$ and 0.7 otherwise. A possible evolution for one simulation can be shown in Fig. 1.

Suppose $x_1(t)$ works in $[-5,5]$. If we divide the operation domain of $x_1(t)$ into 10 subdomains, we will have for each domain l

$$\begin{cases}\overline{x}_{1_l} & =-5+l\\ \underline{x}_{1_l} & =-6+l, l=1,2\ldots,10.\end{cases}$$

TABLE II
MFs OF THE PROPOSED ASYNCHRONOUS FUZZY CONTROLLER

$\overline{\varpi}_1(\mathbf{x}(t)) = \begin{cases} 1 & x_1(t) < -4.8 \\ \frac{-x_1(t)+5.2}{10} & -4.8 \le x_1(t) \le 5.2 \\ 0 & x_1(t) > 5.2 \end{cases}$ $\underline{\varpi}_2(\mathbf{x}(t)) = 1 - \overline{\varpi}_1(\mathbf{x}(t))$
$\underline{\varpi}_1(\mathbf{x}(t)) = \begin{cases} 1 & x_1(t) < -5.2 \\ \frac{-x_1(t)+4.8}{10} & -5.2 \le x_1(t) \le 4.8 \\ 0 & x_1(t) > 4.8 \end{cases}$ $\overline{\varpi}_2(\mathbf{x}(t)) = 1 - \underline{\varpi}_1(\mathbf{x}(t))$

Corresponding to domain l, set $\varsigma_{12\,l}(x_1) = \frac{x_1 - \underline{x}_{1_l}}{\overline{x}_{1_l} - \underline{x}_{1_l}}$ and $\varsigma_{11\,l}(x_1) = 1 - \varsigma_{12\,l}(x_1)$. The constant scalars are $\underline{\eta}_{ij1l} = \underline{\omega}_i(\underline{x}_{1_l})\underline{\varpi}_j(\underline{x}_{1_l})$, $\underline{\eta}_{ij2l} = \underline{\omega}_i(\overline{x}_{1_l})\underline{\varpi}_j(\overline{x}_{1_l})$, $\overline{\eta}_{ij1l} = \overline{\omega}_i(\underline{x}_{1_l})\overline{\varpi}_j(\underline{x}_{1_l})$, $\overline{\eta}_{ij1l} = \overline{\omega}_i(\overline{x}_{1_l})\overline{\varpi}_j(\overline{x}_{1_l})$. When x_1 is not in the domain l, $\varsigma_{11\,l}(x_1) = \varsigma_{12\,l}(x_1) = 0$.

Table II presents the MFs predefined for the asynchronous fuzzy controller.

Then, with the aid of Theorem 2, we conduct the stability analysis and controller design. Set $\zeta_1 = \zeta_2 = \zeta_3 = \zeta_4 = \zeta_5 = 0.1$ and $\xi_j^{\iota} = 10^2$. Suppose the controller gain is a polynomial of degree 0. Using the SOSTOOLs, it can be computed that

$$K_1^1(\mathbf{x}(t)) = \begin{bmatrix} -3.4011 & 10.4709 \end{bmatrix}$$
$$K_2^1(\mathbf{x}(t)) = \begin{bmatrix} -1.2625 & 2.8103 \end{bmatrix}$$
$$K_1^2(\mathbf{x}(t)) = \begin{bmatrix} -3.3971 & 10.4663 \end{bmatrix}$$
$$K_2^2(\mathbf{x}(t)) = \begin{bmatrix} -1.26 & 2.8096 \end{bmatrix} \tag{46}$$

with the noise attenuation level index $\gamma_{\min} = 2.1063$. If we set the controller gain to be a polynomial of degree 2, we can obtain that

$$\begin{aligned} K_1^1(\mathbf{x}) = [&-0.1433x_1^2 + 0.7810x_1 - 5.2171 \\ &1.031x_1^2 - 2.3881x_1 + 22.4773] \\ K_2^1(\mathbf{x}) = [&-0.2220x_1^2 - 0.0933x_1 - 1.0586 \\ &1.1923x_1^2 + 0.2805x_1 + 1.9835] \\ K_1^2(\mathbf{x}) = [&-0.1433x_1^2 + 0.7808x_1 - 5.2111 \\ &1.03x_1^2 - 2.3891x_1 + 22.4816] \\ K_2^2(\mathbf{x}) = [&-0.2219x_1^2 - 0.0933x_1 - 1.0559 \\ &1.1922x_1^2 + 0.2815x_1 + 1.9811] \end{aligned} \tag{47}$$

with $\gamma_{\min} = 2.6348$. To visualize our simulations, we inject the uncertainties into the MFs and here we choose

$$\omega_1(\mathbf{x}(t)) = \varepsilon_1(\mathbf{x}(t))\overline{\omega}_1(\mathbf{x}(t)) + (1 - \varepsilon_1(\mathbf{x}(t)))\underline{\omega}_1(\mathbf{x}(t))$$
$$\omega_3(\mathbf{x}(t)) = \varepsilon_3(\mathbf{x}(t))\overline{\omega}_3(\mathbf{x}(t)) + (1 - \varepsilon_3(\mathbf{x}(t)))\underline{\omega}_3(\mathbf{x}(t))$$
$$\omega_2(\mathbf{x}(t)) = 1 - \omega_1(\mathbf{x}(t)) - \omega_3(\mathbf{x}(t))$$

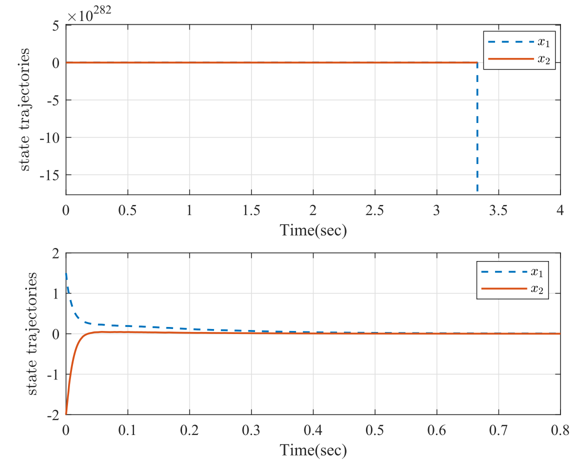

Fig. 2. Trajectories of the states with and without controller.

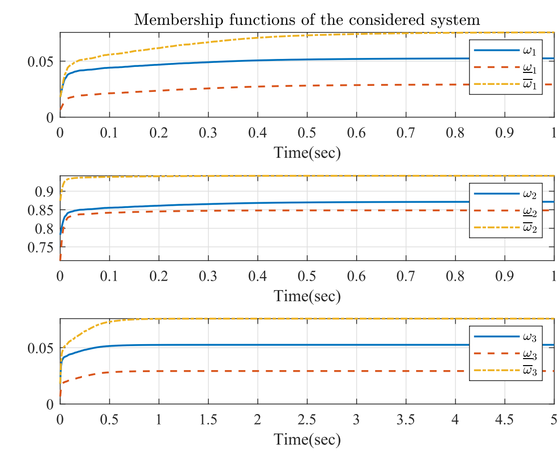

Fig. 3. Trajectories of the MFs.

where $\varepsilon_1(\mathbf{x}(t)) = \frac{1-\sin(x_1(t))}{2}$ and $\varepsilon_3(\mathbf{x}(t)) = \frac{1-\cos(x_1(t))}{2}$. Let the exogenous disturbance $\mathbf{w}(t) = \frac{\sin(0.05t)}{1+t^2}$. Then, with the controller gain (46), the state trajectories can be obtained as is shown in Fig. 2. The unforced system is unstable and is stabilized by the controller designed using our approach. The corresponding MFs can be illustrated by Fig. 3.

Example 2: To further show the application potential of our results, we now consider the following single-link robot arm model presented in [35]

$$J\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} = -MgL\sin(y) - D(t)\frac{\mathrm{d}y}{\mathrm{d}t} + u(t) \tag{48}$$

where the details of the symbols are presented in Table III. Note that the mass of the pay load is assumed to be uncertain and bounded in $[1, 2]$kg. The moment of the inertial has two operation mode with the transitions being the same as in (45).

Now let $x_1(t) = y$ and $x_2(t) = \dot{y}$. Define $f(x_1(t)) = -M \cdot \frac{\sin(x_1(t))}{x_1(t)}$. As discussed in [36], the boundary of f corresponding to the uncertainties in M can be computed as

$$f_{\min}(x_1(t)) = 0.2994x_1^2(t) - 1.9917$$
$$f_{\max}(x_1(t)) = 0.1497x_1^2(t) - 0.9958$$

TABLE III
DETAILS OF THE SYMBOLS

Symbols	description	Value
y	Angular displacement of the arm	—
$u(t)$	Control input	—
$D(t)$	Coefficient of viscous friction	$D_0 = 2$
g	Acceleration of gravity	$9.8\,\mathrm{m/s^2}$
L	Length of the arm	$0.5m$
M	Mass of the pay load	$[1, 2]kg$
J	Moment of inertial	$J_1 = 1kg \cdot m^2$ $J_2 = 2.5kg \cdot m^2$

TABLE IV
UPPER AND LOWER MFS FOR THE SINGLE-LINK ROBOT ARM MODEL

$\underline{\omega}_1(\mathbf{x}(t)) = \frac{f_{\max} - f(x_1(t))}{f_{\max} - f_{\min}}$ $\overline{\omega}_2(\mathbf{x}(t)) = \frac{f(x_1(t)) - f_{\min}}{f_{\max} - f_{\min}}$	with $M = 1$ kg
$\overline{\omega}_1(\mathbf{x}(t)) = \frac{f_{\max} - f(x_1(t))}{f_{\max} - f_{\min}}$ $\underline{\omega}_2(\mathbf{x}(t)) = \frac{f(x_1(t)) - f_{\min}}{f_{\max} - f_{\min}}$	with $M = 2$ kg

TABLE V
MFS FOR PROPOSED ASYNCHRONOUS FUZZY CONTROLLER

$$\overline{\varpi}_1(\mathbf{x}(t)) = \begin{cases} 1 & x_1(t) < -\pi/2 \\ \frac{-x_1(t)+\pi/2}{\pi} & -\pi/2 \le x_1(t) \le \pi/2 \\ 0 & x_1(t) > \pi/2 \end{cases}$$

$$\underline{\varpi}_2(\mathbf{x}(t)) = 1 - \overline{\varpi}_1(\mathbf{x}(t))$$

$$\underline{\varpi}_1(\mathbf{x}(t)) = \begin{cases} 1 & x_1(t) < -\pi/2 \\ \frac{0.9(-x_1(t)+\pi/2)}{\pi} & -\pi/2 \le x_1(t) \le \pi/2 \\ 0 & x_1(t) > \pi/2 \end{cases}$$

$$\overline{\varpi}_2(\mathbf{x}(t)) = 1 - \underline{\varpi}_1(\mathbf{x}(t))$$

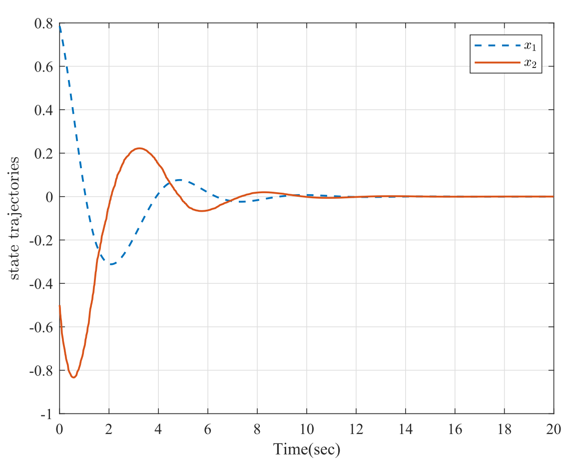

Fig. 4. Trajectories of the states.

following the Taylor series-based method. Then, the system model (5) with the following parameters can be used to represent (48):

$$\mathbf{A}_1^1(\mathbf{x}(t)) = \begin{bmatrix} 0 & 1 \\ f_{\min}(x_1(t)) \cdot gL & -D_0 \end{bmatrix}$$

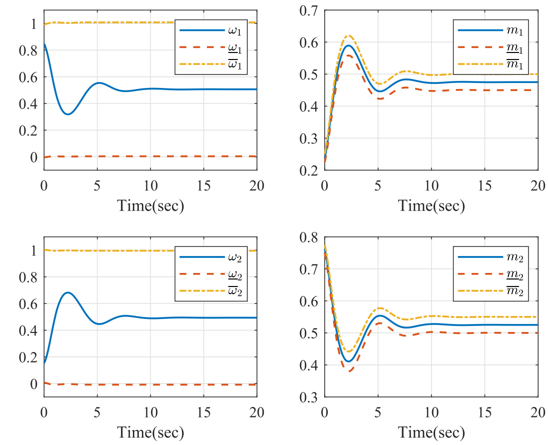

Fig. 5. Trajectories of MFs.

$$\mathbf{A}_2^1(\mathbf{x}(t)) = \begin{bmatrix} 0 & 1 \\ f_{\max}(x_1(t)) \cdot gL & -D_0 \end{bmatrix}$$

$$\mathbf{A}_1^2(\mathbf{x}(t)) = \begin{bmatrix} 0 & 1 \\ f_{\min}(x_1(t)) \cdot 0.4gL & -0.4D_0 \end{bmatrix}$$

$$\mathbf{A}_2^2(\mathbf{x}(t)) = \begin{bmatrix} 0 & 1 \\ f_{\max}(x_1(t)) \cdot 0.4gL & -0.4D_0 \end{bmatrix}$$

$$\mathbf{B}_i^1(\mathbf{x}(t)) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \ \mathbf{B}_i^2(\mathbf{x}(t)) = \begin{bmatrix} 0 \\ 0.4 \end{bmatrix}$$

$$\mathbf{C}_i^p(\mathbf{x}(t)) = \begin{bmatrix} 1 & 0 \end{bmatrix}, \ \mathbf{D}_i^p(\mathbf{x}(t)) = \begin{bmatrix} 0.5 & -1.8 \end{bmatrix}^T$$
$$p = 1, 2, \quad i = 1, 2.$$

The upper and lower MFs are chosen as in Table IV. For implementation purpose, the MFs of the designed controller are set to be linear in the operation range, which is supposed to be $x_1(t) \in [-\pi/2, \pi/2]$. The details are shown in Table V. Here, we equally dive the operation domain of x_1 into five subdomains with $\overline{x}_{1_l} = -\pi/2 + \pi/5 * l$ and $\underline{x}_{1_l} = -7\pi/10 + \pi/5 * l$, $l = 1, 2, 3, 4, 5$. The mode transitions and the rest of the parameters are the same with those in Example 1. Then, by SOSTOOLs the conditions in Theorem 2 can be readily checked and we can obtain $\gamma_{\min} = 4.9939$ and the asynchronous controller gain as

$$K_1^1(\mathbf{x}(t)) = \begin{bmatrix} 4.315 & 0.3332 \end{bmatrix}$$

$$K_2^1(\mathbf{x}(t)) = \begin{bmatrix} 4.38 & 0.2968 \end{bmatrix}$$

$$K_1^2(\mathbf{x}(t)) = \begin{bmatrix} 4.3291 & 0.3956 \end{bmatrix}$$

$$K_2^2(\mathbf{x}(t)) = \begin{bmatrix} 4.3911 & 0.3532 \end{bmatrix}.$$

To further illustrate the results, let

$$\omega_1(\mathbf{x}(t)) = \varepsilon_1(\mathbf{x}(t))\overline{\omega}_1(\mathbf{x}(t)) + (1 - \varepsilon_1(\mathbf{x}(t)))\underline{\omega}_1(\mathbf{x}(t))$$
$$\omega_2(\mathbf{x}(t)) = 1 - \omega_1(\mathbf{x}(t))$$

with $\varepsilon_1(\mathbf{x}(t)) = \frac{1-\sin(x_1(t))}{2}$. The initial value is $x(t) = [\pi/4 - 0.5]^T$. Then, we have the state trajectories as plotted in Fig. 4, which verifies the effectiveness of our results. And Fig. 5 demonstrates the interval type MFs.

V. Conclusion

In this article, the asynchronous control has been analyzed for a class of nonlinear MJSs described by IT2 polynomial fuzzy model. A more general description for the disturbance-induced asynchronous phenomenon has been adopted and well integrated into the synthesis of the IT2 polynomial fuzzy controller for its compact matrix manipulation benefits. The imperfect premise matching problem has also been considered to ease the design and implementation burden of the controllers. Under the framework of SOS approach, feasibility conditions have been obtained by analyzing a proposed joint Markov chain-based functional. The design scheme has been presented in terms of solving a series of SOS conditions, which can be checked by SOSTOOLS. The standard operation of our analysis and synthesis approach makes it ready to be applied in future works concerning asynchronous control issues, such as filtering with multiple constraints, tracking control with networked attacks, and so on.

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